

SOLVABILITY OF A CLASS OF BRAIDED FUSION CATEGORIES

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ABSTRACT. We show that a weakly integral braided fusion category \mathcal{C} such that every simple object of \mathcal{C} has Frobenius-Perron dimension ≤ 2 is solvable. In addition, we prove that such a fusion category is group-theoretical in the extreme case where the universal grading group of \mathcal{C} is trivial.

1. INTRODUCTION AND MAIN RESULTS

Let k be an algebraically closed field of characteristic zero. A fusion category over k is a semisimple tensor category over k having finitely many isomorphism classes of simple objects. In this paper we consider the problem of giving structural results of a fusion category \mathcal{C} under restrictions on the set $\text{c.d.}(\mathcal{C})$ of Frobenius-Perron dimensions of its simple objects.

Results of this type were obtained in the paper [20]. For instance, we showed in [20, Theorem 7.3] that under the assumption that \mathcal{C} is braided odd-dimensional and $\text{c.d.}(\mathcal{C}) \subseteq \{p^m : m \geq 0\}$, where p is a (necessarily odd) prime number, then \mathcal{C} is solvable. Also, the same is true when $\mathcal{C} = \text{Rep } H$, where H is a semisimple quasitriangular Hopf algebra and $\text{c.d.}(\mathcal{C}) = \{1, 2\}$ [20, Theorem 6.12].

Using results of the paper [1], we also showed in [20, Theorem 6.4] that if $\mathcal{C} = \text{Rep } H$, where H is any semisimple Hopf algebra, and $\text{c.d.}(\mathcal{C}) \subseteq \{1, 2\}$, then \mathcal{C} is weakly group-theoretical, and furthermore, it is group-theoretical if \mathcal{C} coincides with the adjoint subcategory \mathcal{C}_{ad} .

Our main results are the following theorems. Recall that a fusion category \mathcal{C} is called *weakly integral* if the Frobenius-Perron dimension of \mathcal{C} is a natural integer.

Theorem 1.1. *Let \mathcal{C} be a weakly integral braided fusion category such that $\text{FPdim } X \leq 2$, for all simple object X of \mathcal{C} . Then \mathcal{C} is solvable.*

Theorem 1.1 extends the previous result for semisimple quasitriangular Hopf algebras mentioned above. It implies in particular that every weakly integral braided fusion category with Frobenius-Perron dimensions of simple

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objects at most 2 is weakly group-theoretical. This gives some further support to the conjecture that every weakly integral fusion category is weakly group-theoretical. See [8, Question 2].

It is known that a nilpotent braided fusion category, which is in addition integral (that is, $\text{c.d.}(\mathcal{C}) \subseteq \mathbb{Z}_+$) is always group-theoretical [4, Theorem 6.10]. We also show that the same conclusion is true in the opposite extreme case:

Theorem 1.2. *Let \mathcal{C} be a weakly integral braided fusion category such that $\text{FPdim } X \leq 2$, for all simple object X of \mathcal{C} . Suppose that the universal grading group of \mathcal{C} is trivial. Then \mathcal{C} is group-theoretical.*

Theorems 1.1 and 1.2 are proved in Section 4. Our proofs rely on the results of Naidu and Rowell [18] for the case where \mathcal{C} is integral and has a faithful self-dual simple object of Frobenius-Perron dimension 2.

Being group-theoretical, a braided fusion category \mathcal{C} satisfying the assumptions of Theorem 1.2, has the so called property **F**, namely, all associated braid group representations on the tensor powers of objects of \mathcal{C} factor over finite groups. See [9, Corollary 4.4]. It is conjectured that every braided weakly integral fusion category does have property **F** [18]. This conjecture has been proved for braided fusion categories \mathcal{C} with $\text{c.d.}(\mathcal{C}) = \{1, 2\}$ such that all objects of \mathcal{C} are self-dual or \mathcal{C} is generated by a self-dual simple object [18, Corollary 4.3 and Remark 4.4].

The paper is organized as follows. In Section 2 we recall the main facts and terminology about fusion and braided fusion categories used throughout. In Section 3 we discuss some families of (integral) examples that appear in the literature. We also recall in this section the results of the paper [18] related to dihedral group fusion rules that will be used later. In Section 4 we give the proofs of Theorems 1.1 and 1.2.

2. PRELIMINARIES

2.1. Fusion categories. Let \mathcal{C} be a fusion category. We shall denote by $\text{Irr}(\mathcal{C})$ the set of isomorphism classes of simple objects of \mathcal{C} and by $G(\mathcal{C})$ the group of isomorphism classes of invertible objects of \mathcal{C} . For an object X of \mathcal{C} , we shall indicate by $\mathcal{C}[X]$ the fusion subcategory generated by X and by $G[X]$ the subgroup of $G(\mathcal{C})$ consisting of invertible objects g such that $g \otimes X \simeq X$.

If \mathcal{D} is another fusion category, \mathcal{C} and \mathcal{D} are *Morita equivalent* if \mathcal{D} is equivalent to the dual $\mathcal{C}_{\mathcal{M}}^*$ with respect to an indecomposable module category \mathcal{M} . Recall that \mathcal{C} is called *pointed* if all its simple objects are invertible and it is called *group-theoretical* if it is Morita equivalent to a pointed fusion category.

There is a canonical faithful grading $\mathcal{C} = \bigoplus_{g \in U(\mathcal{C})} \mathcal{C}_g$, with trivial component $\mathcal{C}_e = \mathcal{C}_{\text{ad}}$, where \mathcal{C}_{ad} is the *adjoint subcategory* of \mathcal{C} , that is, the fusion subcategory generated by $X \otimes X^*$, where X runs through the simple objects of \mathcal{C} . The group $U(\mathcal{C})$ is called the *universal grading group* of \mathcal{C} . \mathcal{C} is called

nilpotent if the upper central series $\cdots \subseteq \mathcal{C}^{(n+1)} \subseteq \mathcal{C}^{(n)} \subseteq \cdots \subseteq \mathcal{C}^{(0)} = \mathcal{C}$ converges to Vec_k , where $\mathcal{C}^{(i)} := (\mathcal{C}^{(i-1)})_{\text{ad}}$, $i \geq 1$. See [11].

A *weakly group-theoretical* fusion category is a fusion category \mathcal{C} which is Morita equivalent to a nilpotent fusion category. If \mathcal{C} is Morita equivalent to a cyclically nilpotent fusion category, then \mathcal{C} is called *solvable*. We refer the reader to [7, 8] for further definitions and facts about fusion categories.

2.2. Braided fusion categories. Let \mathcal{C} be a *braided* fusion category, that is, \mathcal{C} is equipped with natural isomorphisms $c_{X,Y} : X \otimes Y \rightarrow Y \otimes X$, $X, Y \in \mathcal{C}$, satisfying the hexagon axioms. Recall that \mathcal{C} is called *premodular* if it is also spherical, that is, \mathcal{C} has a pivotal structure such that left and right categorical dimensions coincide. Equivalently, \mathcal{C} is premodular if it is endowed with a compatible ribbon structure [2, 16].

We say that the objects X and Y of a braided fusion category \mathcal{C} centralize each other if $c_{Y,X}c_{X,Y} = \text{id}_{X \otimes Y}$. The *centralizer* \mathcal{D}' of a fusion subcategory $\mathcal{D} \subseteq \mathcal{C}$ is defined to be the full subcategory of objects of \mathcal{C} that centralize every object of \mathcal{D} . The centralizer \mathcal{D}' results a fusion subcategory of \mathcal{C} .

The *Müger (or symmetric) center* $Z_2(\mathcal{C})$ of \mathcal{C} is $Z_2(\mathcal{C}) = \mathcal{C}'$; this is a symmetric fusion subcategory of \mathcal{C} whose objects are called central, degenerate or transparent. A braided fusion category \mathcal{C} is called *non-degenerate* if its Müger center $Z_2(\mathcal{C})$ is trivial. A *modular* category is a non-degenerate premodular category \mathcal{C} .

Remark 2.1. Recall that a fusion category \mathcal{C} is called pseudo-unitary if $\dim \mathcal{C} = \text{FPdim } \mathcal{C}$, where $\dim \mathcal{C}$ is the global dimension of \mathcal{C} and $\text{FPdim } \mathcal{C}$ is the Frobenius-Perron dimension of \mathcal{C} . If \mathcal{C} pseudo-unitary then \mathcal{C} has a canonical spherical structure with respect to which categorical dimensions of all simple objects coincide with their Frobenius-Perron dimensions [7, Proposition 8.23].

In particular, this holds for any weakly integral fusion category, because it is automatically pseudo-unitary [7, Proposition 8.24]. Hence every weakly integral non-degenerate fusion category is canonically a modular category.

3. SOME FAMILIES OF EXAMPLES

3.1. Examples of fusion categories with Frobenius-Perron dimensions ≤ 2 . In this subsection we discuss examples of weakly integral fusion categories with Frobenius-Perron dimensions of simple objects ≤ 2 that appear in the literature.

Example 3.1. Consider a Hopf algebra H fitting into an abelian exact sequence:

$$(3.1) \quad k \rightarrow k^\Gamma \rightarrow H \rightarrow k\mathbb{Z}_2 \rightarrow k,$$

where Γ is a finite group. Let $\mathcal{C} = \text{Rep } H$. Then $\text{c.d.}(\mathcal{C}) \subseteq \{1, 2\}$ and equality holds if the associated action of \mathbb{Z}_2 on Γ is not trivial.

All these examples are group-theoretical, in view of [19, Theorem 1.3]. Observe that, as a consequence of [1, Theorem 6.4], any cosemisimple Hopf algebra H such that $\text{c.d.}(\mathcal{C}) \subseteq \{1, 2\}$ is group-theoretical if $\mathcal{C} = \mathcal{C}_{\text{ad}}$. See [20, Theorem 6.4].

Non-trivial examples of cosemisimple Hopf algebras fitting into an exact sequence (3.1) are given by the Hopf algebras

$$\mathcal{A}_{4m}^*, \mathcal{B}_{4m}^* \quad m \geq 2,$$

of dimension $4m$, due to Masuoka [14]. In these cases, Γ is a dihedral group.

Example 3.2. Let $\mathcal{C} = \mathcal{TV}(G, \chi, \tau)$ be the Tambara-Yamagami category associated to a finite (necessarily abelian) group G , a symmetric non-degenerate bicharacter $\chi : G \times G \rightarrow k^\times$ and an element $\tau \in k$ satisfying $|G|\tau^2 = 1$ [24]. This is a fusion category with isomorphism classes of simple objects parameterized by the set $G \cup \{X\}$, where $X \notin G$, obeying the fusion rules

$$(3.2) \quad g \otimes h = gh, \quad g, h \in G, \quad X \otimes X = \bigoplus_{g \in G} g.$$

We have $\text{c.d.}(\mathcal{C}) = \{1, 2\}$ if and only if G is of order 4. Therefore, in this case $\text{FPdim } \mathcal{C} = 8$.

If $G \simeq \mathbb{Z}_4$, there are two possible fusion categories \mathcal{C} . None of them is braided [22, Theorem 1.2 (1)].

If $G \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ there are exactly four classes of Tambara-Yamagami categories with irreducibles degrees 1 or 2, by [24, Theorem 4.1]. Three of them are (equivalent to) the categories of representations of eight-dimensional Hopf algebras: the dihedral group algebra of order 8, the quaternion group algebra, and the Kac-Paljutkin Hopf algebra H_8 . The remaining fusion category, which has the same χ as H_8 but $\tau = -1/2$, is not realized as the fusion category of representations of a Hopf algebra. Since in this case G is an elementary abelian 2-group all of this categories admit a braiding, by [22, Theorem 1.2 (1)].

All the fusion categories in this example are group-theoretical. In fact, by [10, Lemma 4.5], for any symmetric non-degenerate bicharacter $\chi : G \times G \rightarrow k^\times$, G contains a Lagrangian subgroup with respect to χ . Therefore $\mathcal{TV}(G, \chi, \tau)$ is group-theoretical, by [10, Theorem 4.6].

Example 3.3. Recall that a near-group category is a fusion category with exactly one isomorphism class of non-invertible simple object. In the notation of [22], the fusion rules of \mathcal{C} are determined by a pair (G, κ) , where G is the group of invertible objects of \mathcal{C} and κ is a nonnegative integer. Letting $\text{Irr}(\mathcal{C}) = G \cup \{X\}$, where X is non-invertible, we have the relation

$$(3.3) \quad X \otimes X = \bigoplus_{g \in G} g \oplus \kappa X.$$

Near-group categories with fusion rule $(G, 0)$ for some finite group G are thus Tambara-Yamagami categories, discussed in the previous example. Let us consider near-group categories with fusion rule (G, κ) for some finite group G and a positive integer κ .

We have $\text{c.d.}(\mathcal{C}) = \{1, 2\}$ if and only if G is of order 2 and $\kappa = 1$, that means \mathcal{C} is of type $(\mathbb{Z}_2, 1)$. Therefore, in this case $\text{FPdim } \mathcal{C} = 6$ and since $\kappa > 0$, then \mathcal{C} is group-theoretical, by [6, Theorem 1.1]. By [25, Theorem 1.5], there are up to equivalence exactly two non-symmetric braided near-group categories with fusion rule $(\mathbb{Z}_2, 1)$.

Example 3.4. Examples of a weakly integral braided fusion categories which are not integral and Frobenius-Perron dimensions of simple objects are ≤ 2 are given by the Ising categories, studied in [5, Appendix B]. In this case, there is a unique non-invertible simple object X with $X^{\otimes 2} = \mathbf{1} \oplus a$, where a generates the group of invertible objects, isomorphic to \mathbb{Z}_2 (note that these are also Tambara-Yamagami categories). We have here $\text{c.d.}(\mathcal{C}) = \{1, \sqrt{2}\}$ and $\text{FPdim } \mathcal{C} = 4$. Every braided Ising category is modular [5, Corollary B.12].

Other examples come from braided fusion categories with generalized Tambara-Yamagami fusion rules of type (G, \mathbb{Z}_2) , where G is a finite group. See [13]. In these examples, \mathcal{C} is not pointed, the group of invertible objects is G , and $\mathbb{Z}_2 \simeq \Gamma \subseteq G$ is a subgroup such that $X \otimes X^* \simeq \bigoplus_{h \in \Gamma} h$, for all non-invertible object X of \mathcal{C} . Hence we also have $\text{c.d.}(\mathcal{C}) = \{1, \sqrt{2}\}$.

Since they are not integral, these examples are not group-theoretical.

Example 3.5. Let \mathcal{C} be a braided group-theoretical fusion category. Then \mathcal{C} is an equivariantization of a pointed fusion category, that is, $\mathcal{C} \simeq \mathcal{D}^G$, where \mathcal{D} is a pointed fusion category and G is a finite group acting on \mathcal{D} by tensor autoequivalences [17]. In this case, \mathcal{C} contains the category $\text{Rep } G$ of finite-dimensional representations of G as a fusion subcategory.

Suppose that $\text{c.d.}(\mathcal{C}) = \{1, p\}$, where p is any prime number. Then also $\text{c.d.}(G) \subseteq \{1, p\}$. In particular, the group G must have a normal abelian p -complement; moreover, either G contains an abelian normal subgroup of index p or the center $Z(G)$ has index p^3 . See [12, Theorems 6.9, 12.11].

3.2. Fusion rules of dihedral type. Let D_n be the dihedral group of order $2n$, $n \geq 1$. Recall that D_n has a presentation by generators t, z and relations $t^2 = 1 = z^n$, $tz = z^{-1}t$.

The following proposition describes the fusion rules of $\text{Rep } D_n$ (c.f. [14]).

Proposition 3.6. (1) *Suppose n is odd. Then the isomorphism classes of simple objects of $\text{Rep } D_n$ are represented by 2 invertible objects, $\mathbf{1}$ and g , and $r = (n-1)/2$ simple objects X_1, \dots, X_r , of dimension 2, such that*

$$g \otimes X_i = X_i = X_i \otimes g, \quad \forall i = 1, \dots, r,$$

$$X_i \otimes X_j = \begin{cases} X_{i+j} \oplus X_{|i-j|}, & \text{if } i+j \leq r, \\ X_{n-(i+j)} \oplus X_{|i-j|}, & \text{if } i+j > r; \end{cases}$$

where $X_0 = \mathbf{1} \oplus g$.

(2) *Suppose n is even, that is $n = 2m$. Then the isomorphism classes of simple objects of $\text{Rep } D_n$ are represented by 4 invertible objects, $\mathbf{1}$,*

$g, h, f = gh$, and $m - 1$ simple objects X_1, \dots, X_{m-1} , of dimension 2, such that

$$\begin{aligned} g \otimes X_i &= X_i = X_i \otimes g, & \forall i = 1, \dots, m-1, \\ h \otimes X_i &= X_{m-i} = X_i \otimes h, & \forall i = 1, \dots, m-1, \\ X_i \otimes X_j &= \begin{cases} X_{i+j} \oplus X_{|i-j|}, & \text{if } i+j \leq m, \\ X_{2m-(i+j)} \oplus X_{|i+j|}, & \text{if } i+j > m; \end{cases} \end{aligned}$$

where $X_0 = \mathbf{1} \oplus g$ and $X_m = h \oplus f$.

In particular, the group of invertible objects in $\text{Rep } D_n$ is isomorphic to \mathbb{Z}_2 if n is odd, and to $\mathbb{Z}_2 \times \mathbb{Z}_2$ if n is even.

Remark 3.7. Suppose that 4 divides $n = 2m$. Then $X_{m/2}$ is fixed under (left and right) multiplication by all invertible objects of $\text{Rep } D_n$.

Let \mathcal{C} be a fusion category with $\text{c.d.}(\mathcal{C}) = \{1, 2\}$. Suppose that the Grothendieck ring of \mathcal{C} is commutative (for example, this is the case if \mathcal{C} is braided). Assume in addition that the following conditions hold:

- (a) All objects are self-dual, that is $X \simeq X^*$, for every object X of \mathcal{C} .
- (b) \mathcal{C} has a faithful simple object.

Then, it is shown in [18, Theorem 4.2] that \mathcal{C} is Grothendieck equivalent to $\text{Rep } D_n$. Moreover, \mathcal{C} is necessarily group-theoretical.

It is possible to remove the assumption that all the objects are self-dual but it is still necessary the condition of self-duality on the faithful simple object. Namely, suppose that \mathcal{C} is not self-dual, but satisfies

- (b') \mathcal{C} has a faithful self-dual simple object.

In this case \mathcal{C} is still group-theoretical and it is Grothendieck equivalent to $\text{Rep } \tilde{D}_n$, n odd. See [18, Remark 4.4]. Here \tilde{D}_n is the generalized quaternion (binary dihedral) group of order $4n$, that is, the group presented by generators a, s , with relations $a^{2n} = 1, s^2 = a^n, s^{-1}as = a^{-1}$. (Observe that for n odd, \tilde{D}_n is isomorphic to the semidirect product $\mathbb{Z}_n \rtimes \mathbb{Z}_4$, with respect to the action given by inversion, considered in [18]. For even n , $\text{Rep } \tilde{D}_n$ is Grothendieck equivalent to $\text{Rep } D_{2n}$, while $\mathbb{Z}_n \rtimes \mathbb{Z}_4$ has no faithful representation of degree 2.)

Lemma 3.8. *Let $n \geq 2$. Then $(\text{Rep } \tilde{D}_n)_{\text{ad}} = \text{Rep } D_n$. In addition,*

$$(\text{Rep } D_n)_{\text{ad}} = \begin{cases} \text{Rep } D_{n/2}, & \text{if } n \text{ is even,} \\ \text{Rep } D_n, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Recall that when $\mathcal{C} = \text{Rep } G$, where G is a finite group, then $\mathcal{C}_{\text{ad}} = \text{Rep } G/Z(G)$ [11]. The first claim follows from the fact that the center of \tilde{D}_n equals $\{1, s^2\} \simeq \mathbb{Z}_2$. On the other hand, the center $Z(D_n)$ is trivial if n is odd, and equals $\{1, z^{n/2}\} \simeq \mathbb{Z}_2$ if n is even. This implies the second claim and finishes the proof of the lemma. \square

4. PROOF OF THE MAIN RESULTS

In this section we shall prove Theorems 1.1 and 1.2.

Proposition 4.1. *Let \mathcal{C} be a premodular fusion category. Suppose \mathcal{C} has an invertible object g of order n and a simple object X such that*

$$(4.1) \quad g \otimes X = X, \text{ and}$$

$$(4.2) \quad g \text{ centralizes } X.$$

Then we have

- (i) \mathcal{C} is an equivariantization by the cyclic group \mathbb{Z}_n of a fusion category $\tilde{\mathcal{C}}$.
- (ii) If $g \in \mathcal{C}'$, then $\tilde{\mathcal{C}}$ is braided.

Proof. Condition (4.1) ensures the existence of a fiber functor on the fusion category $\mathcal{C}[g]$ generated by g . Then $\mathcal{C}[g]$ is equivalent to $\text{Rep } \mathbb{Z}_n$ as fusion categories.

Moreover, they are equivalent as braided fusion categories. Indeed, (4.1) implies $\mathcal{C}[g] \subseteq \mathcal{C}[X]$ and therefore $\mathcal{C}[g] \subseteq Z_2(\mathcal{C}[X])$, by (4.2). Hence $\mathcal{C}[g]$ is symmetric. Then the only possible twists in \mathcal{C} are $\theta_h = 1$ and $\theta_h = -1$ for all $h \in \langle g \rangle$. But θ_h is not equal to -1 since h centralizes X and $h \otimes X = X$ [15, Lemma 5.4]. Then $\theta_h = 1$ for all $h \in \langle g \rangle$. Therefore $\mathcal{C}[g] \simeq \text{Rep } \mathbb{Z}_n$ as braided fusion categories, as claimed.

Let $\Gamma = \langle g \rangle \subseteq G(\mathcal{C})$. It follows from [5, Theorem 4.18 (i)] that the de-equivariantization $\tilde{\mathcal{C}} = \mathcal{C}_\Gamma$ of \mathcal{C} by Γ is a fusion category and there is a canonical equivalence $\mathcal{C} \simeq \tilde{\mathcal{C}}^\Gamma$ between the category \mathcal{C} and the Γ -equivariantization of $\tilde{\mathcal{C}}$, which shows (i).

Furthermore, if $g \in \mathcal{C}'$ then $\tilde{\mathcal{C}}$ is braided and the equivalence $\mathcal{C} \simeq \tilde{\mathcal{C}}^\Gamma$ is of braided fusion categories [2, 15] (see also [5, Theorem 4.18 (ii)]). Thus we get (ii). This proves the proposition. \square

Lemma 4.2. *Let \mathcal{C} be a fusion category with commutative Grothendieck ring. Suppose that $\mathcal{C} = \mathcal{C}_{\text{ad}}$. If $\mathcal{D}_1, \dots, \mathcal{D}_s$ are fusion subcategories that generate \mathcal{C} as a fusion category, then $\mathcal{D}_1^{(m)}, \dots, \mathcal{D}_s^{(m)}$ generate \mathcal{C} as a fusion category, $\forall m \geq 0$.*

Proof. Since $\mathcal{D}_1, \dots, \mathcal{D}_s$ generate \mathcal{C} , then $(\mathcal{D}_1)_{\text{ad}}, \dots, (\mathcal{D}_s)_{\text{ad}}$ generate \mathcal{C} . In fact, let X be a simple object of \mathcal{C} . There exist simple objects X_{i_1}, \dots, X_{i_t} , with $X_{i_l} \in \mathcal{D}_{i_l}$, $1 \leq i_1, \dots, i_t \leq s$, such that X is a direct summand of $X_{i_1} \otimes \dots \otimes X_{i_t}$. Then $X \otimes X^*$ is a direct summand of

$$X_{i_1} \otimes \dots \otimes X_{i_t} \otimes X_{i_t}^* \otimes \dots \otimes X_{i_1}^* \simeq (X_{i_1} \otimes X_{i_1}^*) \otimes \dots \otimes (X_{i_t} \otimes X_{i_t}^*),$$

where we have used that \mathcal{C} has a commutative Grothendieck ring. Notice that the object in the right hand side belongs to the fusion subcategory generated by $(\mathcal{D}_1)_{\text{ad}}, \dots, (\mathcal{D}_s)_{\text{ad}}$. Since X was arbitrary, it follows that $(\mathcal{D}_1)_{\text{ad}}, \dots, (\mathcal{D}_s)_{\text{ad}}$ generate \mathcal{C}_{ad} . But $\mathcal{C} = \mathcal{C}_{\text{ad}}$ by assumption, then we have

proved that $(\mathcal{D}_1)_{\text{ad}}, \dots, (\mathcal{D}_s)_{\text{ad}}$ generate \mathcal{C} . The statement follows from this by induction on n , since $\mathcal{D}_j^{(n)} = (\mathcal{D}_j^{(n-1)})_{\text{ad}}$, for all $j = 1, \dots, s$, $n \geq 1$. \square

4.1. Braided fusion categories with irreducible degrees 1 and 2. Throughout this subsection \mathcal{C} is a braided fusion category with $\text{c.d.}(\mathcal{C}) = \{1, 2\}$. We regard \mathcal{C} as a premodular category with respect to its canonical spherical structure. See Remark 2.1.

Remark 4.3. Note that $G[X] \neq \mathbf{1}$, for all X such that $\text{FPdim } X = 2$. Moreover, $|G[X]| = 2$ or 4 . In particular the (abelian) group $G(\mathcal{C})$ is not trivial.

Proposition 4.4. *Let g be a non-trivial invertible object such that $g^2 = 1$ and $\theta_g = 1$. Assume that g generates the Müger center \mathcal{C}' of \mathcal{C} as a fusion category. Then \mathcal{C} is the equivariantization of a modular fusion category $\tilde{\mathcal{C}}$ by the group \mathbb{Z}_2 . Furthermore $\text{c.d.}(\tilde{\mathcal{C}}) \subseteq \{1, 2\}$.*

Proof. By assumption $\mathcal{C}' \simeq \text{Rep } \mathbb{Z}_2$ is tannakian. Then the de-equivariantization $\tilde{\mathcal{C}}$ of \mathcal{C} by \mathcal{C}' is a modular category and there is an action of \mathbb{Z}_2 on $\tilde{\mathcal{C}}$ such that $\mathcal{C} \simeq \tilde{\mathcal{C}}^{\mathbb{Z}_2}$ [2, 15]. Since $\text{c.d.}(\tilde{\mathcal{C}}^{\mathbb{Z}_2}) = \text{c.d.}(\mathcal{C}) = \{1, 2\}$, then $\text{c.d.}(\tilde{\mathcal{C}}) \subseteq \{1, 2\}$, by [8, Proof of Proposition 6.2], [20, Lemma 7.2]. \square

Lemma 4.5. *Suppose that $\mathcal{C} \neq \mathcal{C}_{\text{ad}}$ and \mathcal{C}_{ad} is solvable. Then \mathcal{C} is solvable.*

Proof. Since \mathcal{C} is braided, its universal grading group $U(\mathcal{C})$ is abelian [11, Theorem 6.2]. The category \mathcal{C} is a $U(\mathcal{C})$ -extension of \mathcal{C}_{ad} and an extension of a solvable category by a solvable group is again solvable [8, Proposition 4.5 (i)]. Then \mathcal{C} is solvable, as claimed. \square

Lemma 4.6. *Assume $\mathcal{C} = \mathcal{C}_{\text{ad}}$. Then $\text{FPdim } \mathcal{C}' \geq 2$.*

Proof. Suppose on the contrary that $\text{FPdim } \mathcal{C}' = 1$, that is, \mathcal{C} is modular. Then, by [11, Theorem 6.2], $U(\mathcal{C}) \simeq \widehat{G(\mathcal{C})} \simeq G(\mathcal{C})$. By Remark 4.3, $\mathcal{C}_{\text{ad}} \subsetneq \mathcal{C}$, against the assumption. Hence $\text{FPdim } \mathcal{C}' \geq 2$, as claimed. \square

Lemma 4.7. *Suppose \mathcal{C} is generated by a simple object X such that $X \simeq X^*$ and $\text{FPdim } X = 2$. Then we have*

(i) \mathcal{C} is not modular.

Assume $\mathcal{C} = \mathcal{C}_{\text{ad}}$. Then we have in addition

(ii) *There is a group isomorphism $G(\mathcal{C}) \simeq \mathbb{Z}_2$.*

(iii) $G(\mathcal{C}) \subseteq \mathcal{C}'$.

Proof. By [18, Theorem 4.2; Remark 4.4], \mathcal{C} is Grothendieck equivalent to $\text{Rep } D_n$ or $\text{Rep } \tilde{D}_{2n+1}$, for some $n \geq 1$. Since the universal grading group is a Grothendieck invariant, then in the first case $U(\mathcal{C})$ is isomorphic to \mathbb{Z}_2 if n is even and is trivial if n is odd. But $G(\mathcal{C})$, which is also a Grothendieck invariant, is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ if n is even and is isomorphic to \mathbb{Z}_2 if n is odd, by Proposition 3.2. Then $U(\mathcal{C})$ is not isomorphic to $\widehat{G(\mathcal{C})}$, for any n . Therefore \mathcal{C} is not modular, by [11, Theorem 6.2]. Similarly, if \mathcal{C} is

Grothendieck equivalent to $\text{Rep } \tilde{D}_{2n+1}$, we have $U(\mathcal{C}) \simeq \mathbb{Z}_2$ and $G(\mathcal{C}) \simeq \mathbb{Z}_4$. Hence \mathcal{C} is not modular in this case neither. This shows (i).

Notice that the assumption $\mathcal{C} = \mathcal{C}_{\text{ad}}$ implies that \mathcal{C} is Grothendieck equivalent to $\text{Rep } D_n$, for some n odd. Then (ii) follows immediately from the fusion rules of $\text{Rep } D_n$, with n odd (see Proposition 3.2). Since, by (i), \mathcal{C}' is not trivial, then $G(\mathcal{C}') \neq \mathbf{1}$, because $\text{c.d.}(\mathcal{C}') \subseteq \{1, 2\}$ (c.f. Remark 4.3). By part (i), $G(\mathcal{C}') = G(\mathcal{C})$ and (iii) follows. \square

Remark 4.8. If \mathcal{C} is a fusion category as in Lemma 4.7, then the assumption $\mathcal{C} = \mathcal{C}_{\text{ad}}$ is equivalent to saying that \mathcal{C} is Grothendieck equivalent to $\text{Rep } D_n$, for some $n \geq 1$ odd.

Lemma 4.9. *Suppose that $\mathcal{C} = \mathcal{C}_{\text{ad}}$. Then \mathcal{C} is generated by fusion subcategories $\mathcal{D}_1, \dots, \mathcal{D}_s$, $s \geq 1$, where \mathcal{D}_i is Grothendieck equivalent to $\text{Rep } D_{n_i}$ and n_i is an odd natural number, for all $i = 1, \dots, s$.*

Proof. Let $\mathcal{C} = \mathcal{C}[X_1, \dots, X_s]$ for some simple objects X_1, \dots, X_s . Let $\mathcal{D}_i = \mathcal{C}[X_i]$ be the fusion subcategory generated by X_i , $i = 1, \dots, s$. By Lemma 4.2, $(\mathcal{D}_1)_{\text{ad}}, \dots, (\mathcal{D}_s)_{\text{ad}}$ generate \mathcal{C} as a fusion category. Hence, it is enough to consider only those simple objects X_i whose Frobenius-Perron dimension equals 2 (otherwise, $\text{FPdim } X_i = 1$ and $X_i \otimes X_i^* \simeq \mathbf{1}$).

Moreover, iterating the application of Lemma 4.2, we may further assume that $|G[X_i]| = 2$, for all $i = 1, \dots, s$. Thus we have a decomposition $X_i \otimes X_i^* \simeq \mathbf{1} \oplus g_i \oplus X'_i$, where $G[X_i] = \{\mathbf{1}, g_i\}$ and X'_i is a self-dual simple object of Frobenius-Perron dimension 2. Since $X_i \otimes X_i^*$ generates $(\mathcal{D}_i)_{\text{ad}}$, the above reductions allow us to assume that $\mathcal{D}_i = \mathcal{C}[X_i]$ with X_i simple objects of \mathcal{C} such that $\text{FPdim } X_i = 2$ and $X_i \simeq X_i^*$, $\forall i = 1, \dots, s$.

We claim that we can choose the X_i 's in such a way that $(\mathcal{D}_i)_{\text{ad}} \simeq \mathcal{D}_i$. By [18, Theorem 4.2; Remark 4.4], \mathcal{D}_i is Grothendieck equivalent to $\text{Rep } D_{n_i}$ or to $\text{Rep } \tilde{D}_{2n_i+1}$. Iterating the application of Lemma 4.2 and using Lemma 3.8, we obtain that $\mathcal{C} = \mathcal{C}[\mathcal{D}_1, \dots, \mathcal{D}_s]$, with \mathcal{D}_j a fusion subcategory of \mathcal{C} Grothendieck equivalent to $\text{Rep } D_{n_j}$, n_j odd, for all $j = 1, \dots, s$, as we wanted. \square

4.2. Proof of Theorems 1.1 and 1.2. Let \mathcal{C} be a weakly integral fusion category. It follows from [11, Theorem 3.10] that either \mathcal{C} is integral, or \mathcal{C} is a \mathbb{Z}_2 -extension of a fusion subcategory \mathcal{D} . In particular, if $\mathcal{C} = \mathcal{C}_{\text{ad}}$, then \mathcal{C} is necessarily integral.

Lemma 4.10. *Let \mathcal{C} be fusion category and let X, X' be simple objects of \mathcal{C} . Then the following are equivalent:*

- (i) *The tensor product $X^* \otimes X'$ is simple.*
- (ii) *For every simple object $Y \neq \mathbf{1}$ of \mathcal{C} , either $m(Y, X \otimes X^*) = 0$ or $m(Y, X' \otimes X'^*) = 0$.*

In particular, if $X^ \otimes X'$ is not simple, then $\mathcal{C}[X]_{\text{ad}} \cap \mathcal{C}[X']_{\text{ad}}$ is not trivial.*

Proof. The equivalence between (i) and (ii) is proved in [1, Lemma 6.1] in the case where \mathcal{C} is the category of (co)representations of a semisimple Hopf

algebra. Note that the proof *loc. cit.* works in this more general context as well. \square

Proof of Theorem 1.1. The proof is by induction on $\text{FPdim } \mathcal{C}$. As pointed out at the beginning of this subsection, if \mathcal{C} is not integral, then it is a \mathbb{Z}_2 -extension of a fusion subcategory \mathcal{D} . Since \mathcal{D} also satisfies the assumptions of the theorem, then \mathcal{D} is solvable, by induction. Hence \mathcal{C} is solvable as well.

We may thus assume that \mathcal{C} is integral. Therefore $\text{c.d.}(\mathcal{C}) = \{1, 2\}$ and the results of the previous subsection apply. By Lemma 4.5, we may assume that $\mathcal{C} = \mathcal{C}_{\text{ad}}$. Then it follows from Lemma 4.9 that $\mathcal{C} = \mathcal{C}[\mathcal{D}_1, \dots, \mathcal{D}_s]$, with \mathcal{D}_j Grothendieck equivalent to $\text{Rep } D_{n_j}$, n_j odd, $\forall j = 1, \dots, s$.

By Lemma 4.7, $G(\mathcal{D}_j) = \{1, g_j\}$, $\forall j = 1, \dots, s$. We claim that $g_i = g_j$ $\forall 1 \leq i, j \leq s$. Indeed, let $\mathcal{D}_j = \mathcal{C}[X^{(j)}]$, where $X^{(j)} = X_1^{(j)}$ in the notation of Proposition 3.6. Then we have $(X^{(j)})^{\otimes 2} = 1 \oplus g_j \oplus X_2^{(j)}$. Fix $1 \leq i, j \leq s$. Since \mathcal{C} has no simple objects of Frobenius-Perron dimension 4 then $g_i = g_j$ or $X_2^{(j)} \simeq X_2^{(i)}$, by Lemma 4.10. In the first case we are done. In the second case, we note that $\{1, g_j\} = G[X_2^{(j)}] = G[X_2^{(i)}] = \{1, g_i\}$. Then $g_j = g_i$, as claimed. Let $g = g_j = g_i$.

By Lemma 4.7, $g \in \mathcal{D}'_i$, for all $i = 1, \dots, s$. Since \mathcal{D}_i , $1 \leq i \leq s$, generate \mathcal{C} then $g \in \mathcal{C}'$. It follows from Theorem 4.1 (ii) that \mathcal{C} is the equivariantization by \mathbb{Z}_2 of a braided fusion category $\tilde{\mathcal{C}}$. In particular, $\text{FPdim } \tilde{\mathcal{C}} = \text{FPdim } \mathcal{C}/2$ and $\text{c.d.}(\tilde{\mathcal{C}}) \subseteq \{1, 2\}$, by [8, Proof of Proposition 6.2 (1)], [20, Lemma 7.2]. By inductive hypothesis, $\tilde{\mathcal{C}}$ is solvable. Then \mathcal{C} , being the equivariantization of a solvable fusion category by a solvable group is itself solvable [8, Proposition 4.5 (i)]. \square

Theorem 4.11. *Let \mathcal{C} be a weakly integral braided fusion category that $\text{FPdim } X \leq 2$ for all simple object X of \mathcal{C} . Assume in addition that $\mathcal{C} = \mathcal{C}_{\text{ad}}$. Then \mathcal{C} is tensor Morita equivalent to a pointed fusion category $\mathcal{C}(A \rtimes \mathbb{Z}_2, \tilde{\omega})$, where A is an abelian group endowed with an action of \mathbb{Z}_2 by group automorphisms, and $\tilde{\omega}$ is a certain 3-cocycle on the semidirect product $A \rtimes \mathbb{Z}_2$.*

Proof. The assumption $\mathcal{C} = \mathcal{C}_{\text{ad}}$ implies that \mathcal{C} is integral. Hence we may assume that $\text{c.d.}(\mathcal{C}) = \{1, 2\}$. By Lemma 4.9, \mathcal{C} is generated by fusion subcategories $\mathcal{D}_1, \dots, \mathcal{D}_s$, $s \geq 1$, where \mathcal{D}_i is Grothendieck equivalent to $\text{Rep } D_{n_i}$ and n_i is an odd natural number, for all $i = 1, \dots, s$. Furthermore, as in the proof of Theorem 1.1, the assumption that $\mathcal{C} = \mathcal{C}_{\text{ad}}$ implies that $G(\mathcal{D}_i) = \{1, g\}$, for all $1 \leq i \leq s$, and $\mathcal{C}[g] \simeq \text{Rep } \mathbb{Z}_2$ is a tannakian subcategory of the Müger center \mathcal{C}' . So that $\mathcal{C} \simeq \tilde{\mathcal{C}}^{\mathbb{Z}_2}$ is an equivariantization of a braided fusion category $\tilde{\mathcal{C}}$.

Equivariantization under a group action gives rise to exact sequences of fusion categories [3, Subsection 5.3]. In our situation we have an exact sequence of braided tensor functors

$$(4.3) \quad \text{Rep } \mathbb{Z}_2 \rightarrow \mathcal{C} \xrightarrow{F} \tilde{\mathcal{C}}.$$

In addition, since $\mathcal{C}[g] \subseteq \mathcal{D}_i$, then (4.3) induces by restriction an exact sequence

$$(4.4) \quad \text{Rep } \mathbb{Z}_2 \rightarrow \mathcal{D}_i \rightarrow \tilde{\mathcal{C}}_i,$$

for all $i = 1, \dots, s$, where $\tilde{\mathcal{C}}_i$ is the essential image of \mathcal{D}_i in $\tilde{\mathcal{C}}$ under the functor F . Hence $\tilde{\mathcal{C}}_i$ is a fusion subcategory of $\tilde{\mathcal{C}}$, for all i , and moreover $\tilde{\mathcal{C}}_1, \dots, \tilde{\mathcal{C}}_s$ generate $\tilde{\mathcal{C}}$ as a fusion category. Note in addition that $\text{c.d.}(\tilde{\mathcal{C}}), \text{c.d.}(\tilde{\mathcal{C}}_i) \subseteq \{1, 2\}$, for all $i = 1, \dots, s$. On the other hand, exactness of the sequence (4.4) implies that $2n_i = \text{FPdim } \mathcal{D}_i = 2 \text{FPdim } \tilde{\mathcal{C}}_i$ [3, Proposition 4.10]. Hence $\text{FPdim } \tilde{\mathcal{C}}_i = n_i$ is an odd natural number.

Since $\tilde{\mathcal{C}}_i$ is an integral braided fusion category, then the Frobenius-Perron dimension of every simple object of $\tilde{\mathcal{C}}_i$ divides the Frobenius-Perron dimension of $\tilde{\mathcal{C}}_i$ [8, Theorem 2.11]. Thus we get that $\text{FPdim } Y = 1$, for all $Y \in \text{Irr}(\tilde{\mathcal{C}}_i)$. That is, $\tilde{\mathcal{C}}_i$ is a pointed braided fusion category, for all $i = 1, \dots, s$. Since $\tilde{\mathcal{C}}_1, \dots, \tilde{\mathcal{C}}_s$ generate $\tilde{\mathcal{C}}$ as a fusion category, then $\tilde{\mathcal{C}}$ is also pointed. Therefore $\tilde{\mathcal{C}} \simeq \mathcal{C}(A, \omega)$ as fusion categories, where A is an abelian group and $\omega \in H^3(A, k^\times)$.

Group actions on pointed categories were classified by Tambara [23]. In view of [23, Theorem 4.1] and [21, Proposition 3.2], the fusion category $\mathcal{C} \simeq \tilde{\mathcal{C}}^{\mathbb{Z}_2}$ is tensor Morita equivalent to a pointed category $\mathcal{C}(A \rtimes \mathbb{Z}_2, \tilde{\omega})$, where the semidirect product $A \rtimes \mathbb{Z}_2$ is with respect of the induced action of \mathbb{Z}_2 on the group A of invertible objects of $\tilde{\mathcal{C}}$, and $\tilde{\omega}$ is a certain 3-cocycle on $A \rtimes \mathbb{Z}_2$. \square

Proof of Theorem 1.2. The proof is an immediate consequence of Theorem 4.11. \square

Remark 4.12. Let \mathcal{C} be a braided fusion category such that $\text{c.d.}(\mathcal{C}) = \{1, 2\}$. Suppose that \mathcal{C} is nilpotent. By [4, Theorem 1.1] \mathcal{C} admits a unique decomposition (up to the order of factors) into a tensor product $\mathcal{C}_1 \boxtimes \dots \boxtimes \mathcal{C}_m$, where \mathcal{C}_i are braided fusion categories of Frobenius-Perron dimension $p_i^{m_i}$, for some pairwise distinct prime numbers p_1, \dots, p_m . Then \mathcal{C}_i is an integral braided fusion category, for all $i = 1, \dots, m$, and by [8, Theorem 2.11], we get that \mathcal{C}_i is pointed whenever $p_i > 2$. Hence $\mathcal{C} \simeq \mathcal{C}_1 \boxtimes \mathcal{B}$ as braided fusion categories, where \mathcal{C}_1 is a braided fusion category of Frobenius-Perron dimension 2^m such that $\text{c.d.}(\mathcal{C}_1) = \{1, 2\}$, and \mathcal{B} is a pointed braided fusion category.

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